2023

MATHEMATICS — HONOURS

Paper : CC-6

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integers, set of real numbers and set of complex numbers, respectively.

1. Choose the correct alternative with proper justification (**1 mark** for correct answer and **1 mark** for justification) : (1+1)×10

(a) In a ring $(R, +, \cdot)$, $x^2 = x$, $\forall x \in R \neq \{0\}$. Then which one of the following is true?

(i) R is commutative and char $R = 2$ (ii) R is commutative and char $R \neq 2$

(iii) R is non-commutative and char $R = 2$ (iv) R is non-commutative and char $R \neq 2$.

(b) If $(\mathbb{Z}_n, +, \cdot)$ is an integral domain, then $\phi(n)$ is equal to

(i) n (ii) $n - 1$

(iii) $n - 2$ (iv) 1.

(c) If $f : \mathbb{Z} \to \mathbb{Z}_6$ is defined by $f(n) = [n]$ for all $n \in \mathbb{Z}$, then ker f is equal to

(i) $\mathbb{Z}$ (ii) $2\mathbb{Z}$

(iii) $3\mathbb{Z}$ (iv) $6\mathbb{Z}$.

(d) Let $(D, +, \cdot)$ be a division ring with p elements. Then for all $a \in D$,

(i) $a^p = 0$ (ii) $a^p = a - 1$

(iii) $a^p = a$ (iv) $a^p \neq a^2$.

(e) Find the correct statement from the following :

(i) A field has no ideals. (ii) A field has only two ideals.

(iii) A field has only one ideal. (iv) A field may contain an infinite number of ideals.

(f) Let $a\mathbb{Z}$ and $b\mathbb{Z}$ be two ideals of the ring $\mathbb{Z}$ and $c\mathbb{Z} = a\mathbb{Z} \cap b\mathbb{Z}$. Then

(i) $c = gcd(a, b)$ (ii) $c = a.b$

(iii) $c = a/b$ (iv) $c = lcm(a, b)$.

(g) Let V be a vector space of all real valued functions over the field $\mathbb{R}$. Which of the following is not a subspace of V?

(i) $W_1 = \{f \in V : f(0) = f(1)\}$

(ii) $W_2 = \{f \in V : f(2) = 0\}$

(iii) $W_3 = \{f \in V : f \text{ is a continuous function}\}$

(iv) $W_4 = \{f \in V : f(3) = 1 + f(4)\}$.

(h) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping such that $T(1, 2) = (2, 3)$ and $T(0, 1) = (1, 4)$. Then $T(5, 6)$ is

(i) $(6, -1)$ (ii) $(-6, 1)$

(iii) $(-1, 6)$ (iv) $(1, -6)$.

(i) Let A be a 3×3 real matrix with eigenvalues $2, -2, 1$. Then

(i) $A^2 - 2A$ is non-singular (ii) $A^2 + 2A$ is non-singular

(iii) $A^2 - A$ is non-singular (iv) $A^2 + A$ is non-singular.

(j) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$, then A^{-1} is

(i) $\frac{1}{6}(A + I)$ (ii) $\frac{1}{6}(A + 2I)$

(iii) $\frac{1}{6}(A - I)$ (iv) $\frac{1}{6}(A - 2I)$.

Unit - I

2. Answer *any five* questions :

(a) (i) Prove that any finite integral domain is a field.

(ii) Let R be a ring with unity 1. If $a^2 = a$, $\forall a \in R$, show that $1 - 2a$ is a unit. 3+2

(b) (i) Show that the centre of a ring $(R, +, \cdot)$ is a subring of the ring.

(ii) Show that intersection of all subfields of the field of real numbers $\mathbb{R}$ is the field of rational numbers. 3+2

(c) (i) Show that every subring of the ring $(\mathbb{Z}_n, +, \cdot)$, where n is a positive integer, is an ideal of $\mathbb{Z}_n$.

(ii) Prove that the field of real number $\mathbb{R}$ and the field of complex number $\mathbb{C}$ are not isomorphic. 3+2

(d) (i) Show that the characteristic of a finite ring R divides $|R|$, where $|R|$ denotes the cardinality of R.

(ii) Let R be a ring with unity. Prove that $M_n(R)$, ring of all $n \times n$ matrices has the same characteristic as that of R. 3+2

(e) (i) Let f be an epimorphism of a ring R onto a ring S. Then prove that for every ideal J of S, $f^{-1}(J)$ is an ideal of R containing ker f.

(ii) Prove that an ideal of an ideal of a ring R may not be an ideal of R. 3+2

(f) Let R be a commutative ring with identity, $1 \neq 0$. Prove that a proper ideal P of R is a prime ideal of R if and only if R/P is an integral domain. 5

(g) (i) Show that the ring $2\mathbb{Z}$ is not isomorphic to the ring $5\mathbb{Z}$.

(ii) For any integer $n(>1)$, show that $\mathbb{Z}_n \simeq \mathbb{Z}/_{n\mathbb{Z}}$. 2+3

(h) Let $I = \{a + bi \in \mathbb{Z}[i] : a - b$ is divisible by $2\}$. Show that I is a maximal ideal of $\mathbb{Z}[i]$, the ring of Gaussian integer. Is it a prime ideal? Justify your answer. 4+1

Unit - II

3. Answer *any four* questions :

(a) (i) Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a linearly independent set of generators of a vector space V and T be a proper subset of S. Prove or disprove : T is a basis for V.

(ii) Let $\{\alpha, \beta, \gamma\}$ be a basis for a real vector space V and c is a non-zero real number. Prove that $\{\alpha + c\beta, \beta, \gamma\}$ is a basis for V. 2+3

(b) Show that the set S is a subspace of $\mathbb{R}^3$, where $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0, y = z\}$. Find a basis for S and hence find dim S. 2+2+1

(c) (i) A mapping $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x, y, z) = (-x + y + z, x - y + z, x + y - z, x + y + z)$, for all $(x, y, z) \in \mathbb{R}^3$. Examine if T is linear.

(ii) Let V and W be vector spaces over a field F. If a linear mapping $T : V \to W$ is invertible, then show that $T^{-1} : W \to V$ is also a linear transformation. 3+2

(d) (i) Let $S = \{\alpha, \beta, \gamma\}$ and $T = \{\alpha, \alpha + \beta, \alpha + \beta + \gamma\}$ be two subsets of a real vector space V. Show that $L(S) = L(T)$.

(ii) Let V and W be two vector spaces over a field F and $T : V \to W$ be a linear mapping. If ker $T = \{0\}$ and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V, prove that $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is a basis of Im T. 2+3

(e) Let $\mathbb{R}^3$ denote the three dimensional real vector space. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping which is defined as follows :

$T(x, y, z) = (x + y, y + z, z + x)$, for all $(x, y, z) \in \mathbb{R}^3$.

(i) Determine the dimension of Ker T and Im T.

(ii) Find the matrix associated with T, with respect to the standard ordered basis of $\mathbb{R}^3$. (1+1)+3

(f) (i) Prove that two eigenvectors of a square matrix A over a field F, corresponding to two distinct eigenvalues of A are linearly independent.

(ii) If λ be an eigenvalue of a real orthogonal matrix A, then show that $\frac{1}{\lambda}$ is also an eigenvalue of A. 3+2

(g) (i) If $\{\alpha, \beta, \gamma\}$ is a linearly independent set of vectors in a real vector space V, verify the linear independence of set of vectors $\{\alpha-\beta, \beta-\gamma, \gamma-\alpha\}$.

(ii) Use Cayley-Hamilton theorem to find A^{100}, where $A=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$. 2+3

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